Coherent phenomena in stochastic dynamical systems

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Introduction

Many physical processes take place in the complex media, whose parameters could be viewed as space-time realizations of chaotic fields. Such dynamic problems are too complex to allow an explicit mathematical solution for specific realizations of the media. However, one is often interested in generic features of random solutions, rather than particular details. So one is naturally inclined to adopt the well developed machinery of random fields and processes, that is to replace individual realizations with statistical averages.

Randomness of the medium gives rise to stochastic physical fields. Thus a typical realization of, say 2D scalar fields $\rho(\mathbf{R},t)$ with $\mathbf{R}=(x,y)$ would resemble a complex mountain terrain with randomly distributed peaks, valleys, passes, etc. But the standard statistical tools, like means $\langle\rho(\mathbf{R},t)\rangle$, and moments $\langle\rho(\mathbf{R},t)\,\rho(\mathbf{R}',t')\rangle$, would often smooth out some important qualitative features of individual realizations.

So the resulting "mean fields" would bear little likeness to a typical realization, and sometimes give conflicting predictions. Thus, standard statistical means could reasonably predict some "global" spatial-temporal scales and parameters of solutions, but tell little about the small scale structure and details of evolution. We shall call physical phenomena that occur with probability one and characterize 'typical realizations' *coherent*.

The complete statistics would allow a complete description of the dynamical system. But in practice one could handle only a few simple statistics, typically expressed through the one-point probability distributions (PDF). The natural problem then is to deduce some important qualitative and quantitative characteristics of individual realizations from such limited data.

1 Examples of dynamical systems

1.1 Particles in random velocities and forces

Diffusion of low-inertial particles in random hydrodynamic flow satisfies the Newton equations

$$\begin{aligned}\frac{d}{dt}\mathbf{r}(t) &= \mathbf{V}(t), \quad \mathbf{r}(0)=\mathbf{r}_0,\\ \frac{d}{dt}\mathbf{V}(t) &= -\lambda\left[\mathbf{V}(t)-\mathbf{u}(\mathbf{r}(t),t)\right], \quad \mathbf{V}(0)=\mathbf{V}_0(\mathbf{r}_0).\end{aligned} \tag{1}$$

with the linear friction force described by the Stokes formula $\mathbf{F}(t)=\lambda\mathbf{V}(t)$ for a slowly moving particles, under the effect of random force $\mathbf{f}(t)=\lambda\mathbf{u}(\mathbf{r}(t),t)$ induced by the hydrodynamic flow.

For inertialess particles, the parameter $\lambda\to\infty$ and, as follows from Eqn (1), we arrive at

$$\mathbf{V}(t)=\mathbf{u}(\mathbf{r}(t),t), \tag{2}$$

and the particle's trajectory in a hydrodynamic flow is described by the simplest equation

$$\frac{d}{dt}\mathbf{r}(t)=\mathbf{u}(\mathbf{r}(t),t), \quad \mathbf{r}(0)=\mathbf{r}_0. \tag{3}$$

Thus, the problem of determining trajectories of inertialess particles is a purely kinematic one.

Let us discuss some qualitative features of stochastic system (3) in the absence of the mean flow. Formally equation (3) describes the motion of independent particles, as no interaction takes place. If however, field $\mathbf{u}(\mathbf{r},t)$ has finite correlation radius l_{cor}, then particles within l_{cor} - proximity of each other lie in the common domain of influence of velocity $\mathbf{u}(\mathbf{r},t)$. Hence, they could exhibit a collective behavior.

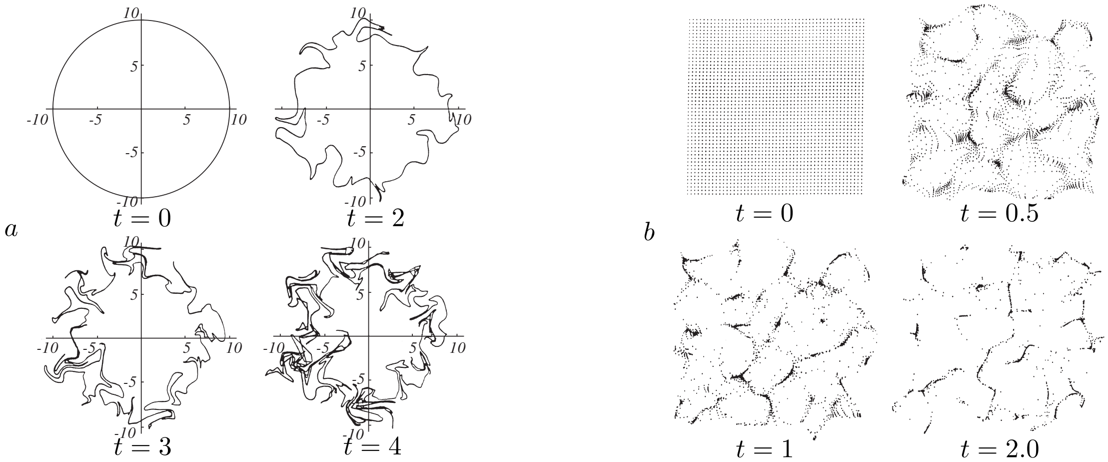

Figure 1: Particle dynamics in solenoidal (a) and potential (b) velocity fields $\mathbf{u}(\mathbf{r})$.

In general, velocity field $\mathbf{u}(\mathbf{r},t)$ is made of the solenoidal ($\operatorname{div}\mathbf{u}(\mathbf{r},t) = 0$), plus potential ($\operatorname{div}\,\mathbf{u}(\mathbf{r},t) \neq 0$) components. Numeric simulations of multiparticle systems, driven by (3) show marked difference between the two cases. Fig. 1a shows a divergent-free random field $\mathbf{u}(\mathbf{r})$ advecting a uniformly distributed set of particles over the disk. Here the total area enclosed by the deformed contours is conserved, and the particles fill the area in a "uniform" manner. Observe, however, that contours become increasingly more rugged and "fractal-like".

In the presence of potential component ($\operatorname{div}\,\mathbf{u}(\mathbf{r},t) \neq 0$), the initial uniform distribution of particles (over the square) evolves into *clusters* - compact regions of high concentration amidst low-density voids. The results of numeric simulations are shown in Fig. 1b. Let us stress here, the *kinematic* nature of this effect. Indeed, the ensemble averaging over velocity realizations could completely obliterate it.

Such clustering of particle systems was first observed, via computer simulation of a simple model of atmospheric dynamic, based on the so called *EOLE experiment*. This global experiment was conducted in Argentina in 1970-71, and involved launching 500 air balloons of constant density, that spread over the entire Southern hemisphere at the altitude roughly 12 km. Fig. 2 shows numeric simulation of the distribution of balloons 105 days after the beginning, and clearly exhibits their convergence into clusterized groups.

The statistical analysis show that *typical realization* of relative displacement of two particles, for example in 2d case is function

$$l^*(t) = l_0 \exp\left\{\frac{1}{4}\left(D^{\mathrm{s}} - D^{\mathrm{p}}\right)t\right\},$$

grows or decreases exponentially in time, depending on sign of ($D^{\mathrm{s}} - D^{\mathrm{p}}$). In particular, incompressible flows ($D^{\mathrm{p}} = 0$) have exponentially increasing typical realizations, which means exponential divergence of particles at short distances and times. At the opposite end stand pure potential velocities ($D^{\mathrm{s}} = 0$). Here a typical realization of particle-separation would exponentially decrease, so the particles tend to coalesce. Such tendency of the flow to "bring particles together" could lead to formation of *clusters*. Indeed, our conclusion is consistent with some numeric studies (illustrated in Fig. 1b), although our model of random velocities is different from those used in computations.

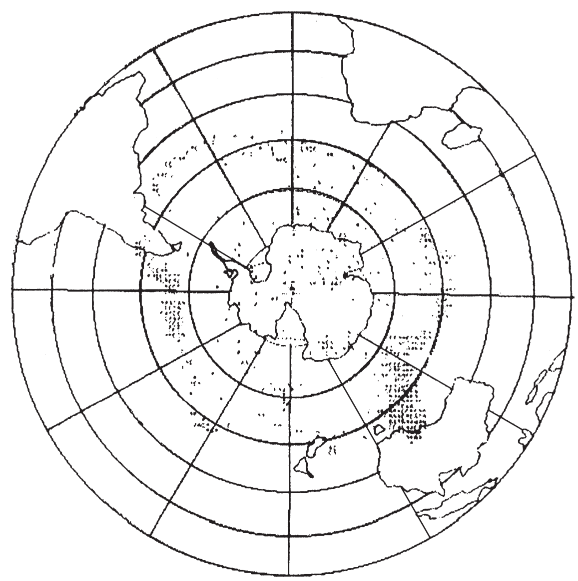

Figure 2: Distribution of air balloons

1.2 Diffusion of density field in random velocity fields

Diffusion of the density field $\rho(\mathbf{r},t)$ of a passive tracer satisfies the continuity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{r}}\mathbf{V}(\mathbf{r},t)\right)\rho(\mathbf{r},t)=0, \qquad \rho(\mathbf{r},0)=\rho_0(\mathbf{r}). \tag{4}$$

Here, $\mathbf{V}(\mathbf{r},t)$ denotes the velocity field of particles in a hydrodynamic flow $\mathbf{u}(\mathbf{r},t)$, which for low-inertia particles can be described by a partial derivative quasi-linear equation

$$\left(\frac{\partial}{\partial t}+\mathbf{V}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V}(\mathbf{r},t)=-\lambda\left[\mathbf{V}(\mathbf{r},t)-\mathbf{u}(\mathbf{r},t)\right]. \tag{5}$$

In the general case, the nonuniqueness of the solution of Eqn (5), discontinuities, etc. are possible. However, in an asymptotic case of low-inertia particles (parameter $\lambda\to\infty$), there exists a unique solution over a reasonable time interval.

The total tracer mass remains unaltered during evolution, i.e., we have

$$M=M(t)=\int d\mathbf{r}\rho(\mathbf{r},t)=\int d\mathbf{r}\rho_0(\mathbf{r})=\text{const}.$$

Given a random field $\mathbf{V}(\mathbf{r},t)$ is Gaussian, statistically homogeneous, spatially isotropic, and steady in time, with a zero mean value, the one-point probability density $P(\mathbf{r},t;\rho)=\langle\delta\left(\rho(\mathbf{r},t)-\rho\right)\rangle$ for the solution of dynamic equation (4) in the approximation of the delta-correlated in time field $\mathbf{V}(\mathbf{r},t)$ is described by equation:

$$\left(\frac{\partial}{\partial t}-D_0\frac{\partial^2}{\partial \mathbf{r}^2}\right)P(\mathbf{r},t;\rho) \quad = \quad D_\rho\frac{\partial^2}{\partial\rho^2}\rho^2P(\mathbf{r},t;\rho),$$

where the diffusion coefficients have the forms (for example in 2D case)

$$\begin{aligned} D_0 &= \frac{1}{2}\int\limits_0^\infty d\tau\left\langle\mathbf{V}(\mathbf{r},t+\tau)\mathbf{V}(\mathbf{r},t)\right\rangle=\frac{1}{2}\tau_{\mathbf{V}}\left\langle\mathbf{V}^2(\mathbf{r},t)\right\rangle, \\ D_\rho &= \int\limits_0^\infty d\tau\left\langle\frac{\partial\mathbf{V}(\mathbf{r},t+\tau)}{\partial\mathbf{r}}\frac{\partial\mathbf{V}(\mathbf{r},t)}{\partial\mathbf{r}}\right\rangle=\tau_{\mathrm{div}\mathbf{V}}\left\langle\left(\frac{\partial\mathbf{V}(\mathbf{r},t)}{\partial\mathbf{r}}\right)^2\right\rangle. \end{aligned}$$

Here the characteristic times $\tau_{\mathbf{V}}$ and $\tau_{\mathrm{div}\mathbf{V}}$ give time correlation radii for random fields $\mathbf{V}(\mathbf{r},t)$ and div $\mathbf{V}(\mathbf{r},t)$.

Notice however, that the diffusion coefficient D_0 gives only global characteristics and scales of the tracer distribution, and carries little information about the fine structure and details of realizations. The diffusion coefficient D_ρ gives information about cluster formation.

For the divergent-free velocities ($D_\rho=0$) the density iso-contours $\rho(\mathbf{r},t)$ =const evolve along the particle trajectories, described in section 1, and illustrated in Fig. 1 *a*. Here the total area bounded by contour $\rho(\mathbf{r},t)=\rho$ =const, and the total mass inside the region, are conserved . Such flows also conserve the number N of ρ-contours. But as evidenced from the Fig. 1 *a*, the contour grows increasingly rugged, with sharpening gradients and evolving small scale structures.

Besides, the average contour length $\rho(\mathbf{r},t)=\rho-$const also grows exponentially as

$$\langle L(t,\rho)\rangle=l_0\exp\left\{D^{\mathrm{s}}t\right\}.$$

Of course, compressible flows with nonzero potential component of $\mathbf{V}(\mathbf{r},t)$, would have both quantities evolve in time. Examples include the *mean area* at large time $\tau\gg 1$, where $\tau=D_\rho t$, enclosed by contours: $\rho\left(\mathbf{r},t\right)\geq\rho$ decreases in time according to:

$$\langle S(t,\rho)\rangle\approx\frac{1}{\sqrt{\pi\tau\rho}}e^{-\tau/4}\int d\mathbf{r}\sqrt{\rho_0(\mathbf{r})},$$

whereas the enclosed mass within the ρ-area

$$\langle M(t,\rho)\rangle\approx M-\sqrt{\frac{\rho}{\pi\tau}}e^{-\tau/4}\int d\mathbf{r}\sqrt{\rho_0(\mathbf{r})}$$

converges monotonically to the total mass of the system. The last result confirms our earlier conclusion regarding *clustering* of tracer in the tightly bounded regions of high density.

1.3 Localization of plane waves in randomly layered media

Let us consider a inhomogeneous layered medium occupying strip $L_0 < x < L$. A plane wave of unit amplitude $u_0(x) = e^{-ik(x-L)}$ is incident upon it from the right half-space $x > L$ (Fig. 3).

The wavefield in the strip obeys the Helmholtz equation

$$\frac{d^2}{dx^2}u(x) + k^2[1+\varepsilon(x)]u(x) = 0, \qquad (6)$$

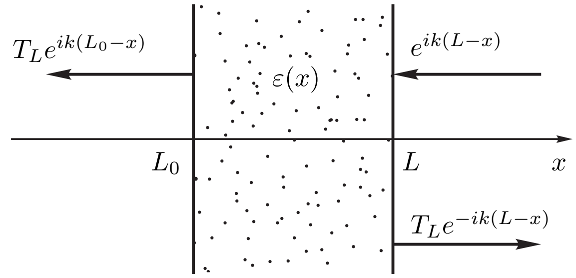

Figure 3: Incident plane wave on a layer

with function $\varepsilon(x)$ representing inhomogeneities of the media. We assume $\varepsilon = 0$ outside the strip, and $\varepsilon(x) = \varepsilon_1(x) + i\gamma$ within, the real part $\varepsilon_1(x)$ responsible for the *wave scattering*, while imaginary one γ describing wave attenuation by the media. The boundary conditions for (6) are continuity relations for $u(x)$ and its derivative $du(x)/dx$ at $x = L;\ L_0$:

$$u(L) + \frac{i}{k}\frac{du(x)}{dx}\bigg|_{x=L} = 2, \qquad u(L_0) - \frac{i}{k}\frac{du(x)}{dx}\bigg|_{x=L_0} = 0. \qquad (7)$$

If parameter $\varepsilon_1(x)$ is random, one is interested in the statistics of the reflection and transmission coefficients: $R_L = u(L) - 1$, and $T_L = u(L_0)$, as well as the field intensity $I(x) = |u(x)|^2$ within the layer (*statistical radiative transport*).

If layer $[L_0, L]$ is sufficiently wide ($\tau = D(L - L_0) \gg 1$), and dissipation-free ($\gamma = 0$), one has the so called *stochastic parametric resonance,* manifested by the initial exponential growth of all moments $\{\langle I^n(x; L)\rangle : n > 1\}$, of the intensity ($I(x; L) = |u(x; L)|^2$) inside the layer. They reach maximum somewhere in the middle of the layer (Fig. 4). In the half-space limit ($L_0 = -\infty$), the range of exponential (explosive) growth extends through the entire half-line, but the mean intensity remains fixed $\langle I(x; L)\rangle = 2$.

The statistical analyze show that all statistical moments of $|T|$ converge to zero with increasing parameter τ ($L_0 \to -\infty$), and we get the reflection modulus $|R| \to 1$ with probability one. Hence randomly stratified half-space is fully reflective. In this case the distribution of wave intensity $I(x)$ is log-normal, having all moments starting with the second one to grow exponentially inside the random layer

$$\langle I^n(L - x)\rangle \sim e^{Dn(n-1)(L-x)},$$

and its typical realization has the form

$$I^*(x) = 2e^{-D(L-x)}.$$

In the physics of disordered systems such exponential fall-off in variable $\xi = D(L - x)$ for a typical realization is associated with the dynamical localization, the *localization length* being $l_{\text{loc}} = 1/D$.

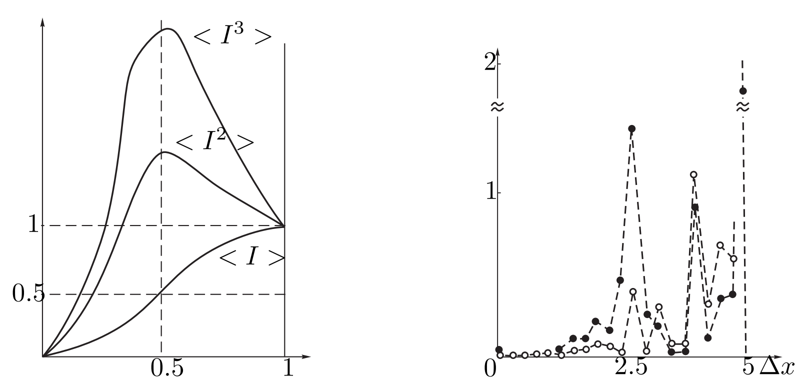

Figure 4: Stochastic parametric resonance and numeric modeling of the dynamic localization for two realizations of random media

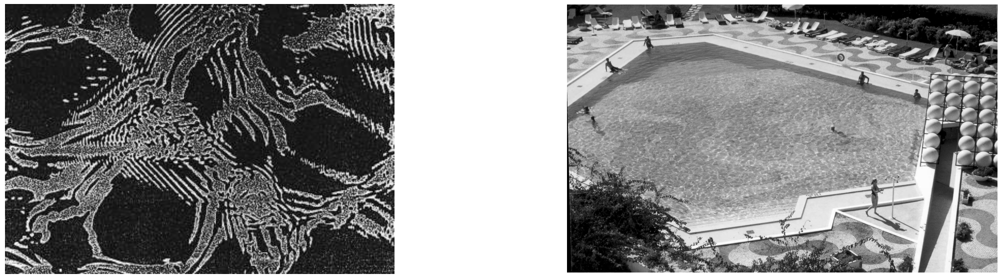

Figure 5: Cross-section of laser beam in turbulent media and caustics in swimming pool

Fig. 4 also shows numeric simulations of two wave intensities in a sufficiently thick layer, that come from two different realizations of the medium. Let us note a clearly perceived exponential fall-off trend accompanied by large intensity fluctuations, directed both ways (to zero and to infinity). They result from the multiple scattering processes in randomly inhomogeneous media, and demonstrate the so called *dynamic localization*.

1.4 Caustical structure of wavefield in random inhomogeneous media

We shall discuss wave propagation in random 2D and 3D media within the so-called scalar *parabolic approximation*. It holds for large scale inhomogeneities and relatively short waves, hence small scattering angles

$$\frac{\partial}{\partial x}U(x,\mathbf{R}) = \frac{i}{2k}\Delta_{\mathbf{R}}U(x,\mathbf{R}) + \frac{ik}{2}\varepsilon(x,\mathbf{R})U(x,\mathbf{R}), \quad U(0,\mathbf{R}) = U_0(\mathbf{R}).$$

Here x denotes the preferred direction of wave propagation, $\mathbf{R}$ - transverse variable, and $\varepsilon(x,\mathbf{R})$ - the deviation of the dielectric permeability from its uniform value 1.

Next we introduce amplitude and phase for wavefield $U(x,\mathbf{R})$

$$u(x,\mathbf{R}) = A(x,\mathbf{R})\exp\{iS(x,\mathbf{R})\},$$

Than the intensity $I(x,\mathbf{R}) = |u(x,\mathbf{R})|^2$ obeys by the transport equation

$$\frac{\partial}{\partial x}I(x,\mathbf{R}) + \frac{1}{k}\boldsymbol{\nabla}_{\mathbf{R}}\{\boldsymbol{\nabla}_{\mathbf{R}}S(x,\mathbf{R})I(x,\mathbf{R})\} = 0, \quad I(0,\mathbf{R}) = I_0(\mathbf{R}). \tag{8}$$

Equation (8) closely resembles (4), and could be viewed as the 'tracer transport' by the potential velocity $\mathbf{u}(x,\mathbf{R}) = \boldsymbol{\nabla}_{\mathbf{R}}S(x,\mathbf{R})$.

As we mentioned earlier the realizations of the intensity field should cluster into the *caustic structures*. Indeed, Fig. 5 shows a cross sectional photograph of the laser beam propagating through the turbulent medium (laboratory experiment) and a swimming pool with the clearly seen caustic structures at the bottom. The latter arises due to the refraction and reflection of light by the perturbed water surface (the so called *phase screen*).

Note that for plane incident wave $U(0,\mathbf{R}) = U_0$−const, the mean value of wavefield is constant ($\langle I(x,\mathbf{R})\rangle = I_0$) and correlation function has the form

$$\langle U(x,\mathbf{R}_1)U^*(x,\mathbf{R}_2)\rangle = I_0 e^{-k^2 D(\mathbf{R}_1-\mathbf{R}_2)x}.$$

In this case the mean wavefield intensity does not depend on an appearance of a diffraction.

2 Statistical topography of random processes and fields

As we mentioned in the introduction solutions of many stochastic dynamical problems exhibit large fluctuations about special deterministic curves, that determine the 'large-scale dynamics' of the system on the entire time-interval. We shall call such curves *typical realizations*, and define them through one-point PDF of the process.

2.1 Typical realizations of random processes

Let $z(t)$ be a random process with one-point PDF and *integral distribution function*

$$p(z;t) = \langle \delta(z(t)-z) \rangle, \ F(z;t) = P(z(t) < z) = \int_{-\infty}^{z} dz' p(z';t). \tag{9}$$

We call a *typical realization* of the process $z(t)$ a deterministic *median* curve of (9) and computed from an algebraic equation

$$F(z^*(t);t) = 1/2.$$

The motivation for this definition comes from the properties of the median. Namely, for any time-interval $[t_1, t_2]$ process $z(t)$ "winds around" the median in such a way that it spends on average half of the time above it, $z(t) > z^*(t)$, and half of time below, $z(t) < z^*(t)$ (see Fig. 6)

$$\langle T_{z(t)>z^*(t)} \rangle = \langle T_{z(t)<z^*(t)} \rangle = \frac{1}{2}(t_2 - t_1).$$

Of course, such $z^*(t)$ would bear little resemblance to any particular realization of the process, and tell nothing about the scope and size of fluctuations. Evidently, typical realization $z^*(t)$ of random process $z(t)$ is well defined on the entire time range $t \in [0, \infty]$.

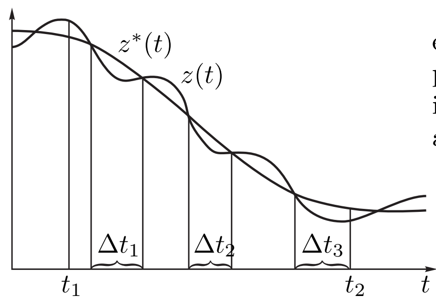

Figure 6: Typical realization of random process

2.2 Statistical topography of random fields

The main subject of statistical topography, like the usual one (i.e. topographic maps of "mountain terrains"), is the set of iso-contours in 2D (or 3D iso-surfaces) of constant field $f - f(\mathbf{R}, t) = f$ =const.

To analyze such contours (for the sake of presentation we shall talk about 2D case) let us introduce a singular indicator function of level f, viewed as a "functional" of the media parameters $\varphi(t, \mathbf{R}; f) = \delta(f(\mathbf{R}, t) - f)$. Such function yield several geometric characteristics of contours. Those include total area, enclosed by $f(\mathbf{R}, t) \geq f$ and total mass inside the region

$$S(t;f) = \int_{f}^{\infty} df' \int d\mathbf{R} \varphi(t, \mathbf{R}; f'), \ M(t;f) = \int_{f}^{\infty} f' df' \int d\mathbf{R} \varphi(t, \mathbf{R}; f').$$

The ensemble average of indicator function gives one-point PDF of the tracer density $P(t, \mathbf{R}; f) = \langle \varphi(t, \mathbf{R}; f) \rangle$. Hence one could also get statistical means of these geometric invariants.

Additional geometric information about density contours could be obtained from values of $f(\mathbf{R}, t)$ combined with its spatial gradient $\mathbf{p}(\mathbf{R}, t) = \nabla f(\mathbf{R}, t)$. For instance, integral

$$l(t;f) = \int d\mathbf{R} \, |\mathbf{p}(\mathbf{R}, t)| \, \delta(f(\mathbf{R}, t) - f) = \oint dl$$

gives the total contour length at level f.

Higher derivatives of $f(\mathbf{R}, t)$ (e.g. second order) furnish an additional geometric information, like the total number of closed contours at a given level $f(\mathbf{R}, t)$ =const. The latter could be approximately expressed (excluding non-closed ones) by the formula

$$N(t;f) = N_{\text{in}}(t;f) - N_{\text{out}}(t;f) = \frac{1}{2\pi} \int d\mathbf{R} \kappa(t, \mathbf{R}; f) \, |\mathbf{p}(\mathbf{R}, t)| \, \delta(f(\mathbf{R}, t) - f),$$

Here $\kappa(t, \mathbf{R}; \rho)$ denotes the curvature along the contour, while $N_{\text{in}}(t, f)$, $N_{\text{out}}(t, f)$ - count contours with the inward or outward pointing gradient $\mathbf{p}(\mathbf{R}, t)$.

2.3 Log-normal processes

We define a log-normal random process

$$y(t;\alpha) = e^{-\alpha t + w(t)} = \exp\left\{-\alpha t + \int\limits_0^t d\tau \xi(\tau)\right\}, \tag{10}$$

in terms of the Gaussian white noise $\xi(t)$:

$$\langle \xi(t) \rangle = 0;\ \langle \xi(t)\xi(t') \rangle = 2D\delta(t-t').$$

It has the following properties

1. Log-normal process is Markovian, and its one-point PDF solves the FP-equation

$$\frac{\partial}{\partial t}P(y,t;\alpha) = \alpha\frac{\partial}{\partial y}yP(y,t;\alpha) + D\frac{\partial}{\partial y}y\frac{\partial}{\partial y}yP(y,t;\alpha), \quad P(y,0;\alpha) = \delta(y-1).$$

   Solution of which has long tail, which shows that one-point statistics of $y\,(t;\alpha)$ are dominated by large deviations.

2. The moments of $y(t;\alpha)$ increase exponentially in time

$$\langle y^n(t;\alpha) \rangle = e^{n(n-\alpha/D)Dt}, \quad \left\langle y^{-n}(t;\alpha) \right\rangle = e^{n(n+\alpha/D)Dt}, \quad n = 1, 2, ...$$

3. Yet the typical realization of the process falls off exponentially $y^*(t;\alpha) = e^{-\alpha t}$. Hence, exponential growth of the moments is due to large fluctuations about $y^*\,(\tau)$ on both sides (large and small values of y).

4. For any probability $0 < p < 1$ there exists a one-parameter family of exponentially decaying curves,

$$M_p(t,\alpha,\beta) = \frac{1}{(1-p)^{D/\beta}} e^{(\beta-\alpha)t}.$$

   that dominate the process in the sense that a sizable fraction of realizations lie below $M_p(t,\alpha)$, i.e. probability $P\left\{y(t;\alpha) < M_p(t,\alpha) \,\text{for all}\, t \in (0,\infty)\right\} = p$.

5. Random variables: $S_n = \int\limits_0^\infty d\tau y^n(\tau)$, that characterize large deviations, have finite (stationary) probability distributions with polynomial fall off at large S:

$$P_n(S) = \frac{n^{-2/n}}{\Gamma(1/n)} \frac{1}{S^{1+1/n}} \exp\left\{-\frac{1}{n^2 S}\right\}.$$

All the above properties manifest in such coherent phenomena as *localization* and *clustering.*

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